AN EPISTEMIC MODEL ANALYSIS FOR LOGIC GAMES

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Abstract. The paper argues in favour of a larger logical setting for critical thinking tests. First it will analyze a solving procedure for a particular case of logical games. Then the paper will introduce an epistemic logic approach to the same game. A Kripke model for the game will be constructed from the information in the game and the game will then be solved in the new logical setting. Some remarks will be made for the new situations originated from the epistemic model proposed, engaging the evaluations of the epistemic states of the agent in terms of belief and knowledge for different situations in the game. Finally, the paper will suggests some developments for critical thinking logical games' setting.

Keywords: logic games, epistemic logic, Kripke model, knowledge, belief.

1. If we could briefly say what critical thinking is about, possibly a single word will be enough to cover the task: arguments. Critical thinking is concerned with arguments, their analysis and evaluation. The literature in the field of argumentation and informal logic has grown constantly since the last half of the XX-th century[1]. The interest in this area is also demonstrated by the proliferation of standardized tests especially built to evaluate the ability to think critical in various situations: GMAT Critical Reasoning Tests, LSAT, California Critical Thinking Tests, etc. Their capacity for selecting individuals with good decision making and problem solving abilities as well as creative thinking abilities seem to determine some institutions to include such tests in the process of selecting their future employes. Each test contains a series of questions and answers relative to a specific critical thinking problem. And a critical thinking problem in the test is intended to test a certain critical thinking ability: analytical reasoning, logical reasoning, or reading comprehension. There are typical solving procedures for each type of question involving these abilities that involve informal logic knowledges. It is our conjecture in this paper that a modal logical approach will deepen the frame of the critical thinking solving procedures and will endow them with new and challenging developments. In what will follow we will present the procedural requests for a case of critical thinking test in a solving scenario, and then we will propose a different epistemic model approach for the case.

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[1] Gabbay, M. Dov, Ralph H. Johnson, Hans Jürgen Ohlbach and John Woods (eds.). *Handbook of the Logic of Argument and Inference The Turn Towards the Practical. Studies in Logic and Practical Reasoning*. Volume 1, Amsterdam, Boston, Heidelberg, London, New York, Oxford, Paris, Elsevier, 2002, pp. 1-36.

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2. We will chose for our purpose here just one particular type of critical ability and one particular type of test: a logic game test for analytical reasoning evaluation in LSAT tests'. For what concerns our research, any other type of test could be treated in a similar way. Any logic game has the following structure: a *fact pattern*, which is the factual context of the logic game, some informations conveying the logical *constrains* of the game, and the questions, each of them with five answers. There is only one correct answer for each question. We will first solve the game within its classical procedural steps recommended to solve the game in the amount of time given (between 77 seconds and 96 seconds per question in a standard LSAT test). Afterwards, we will take a closer look at the dynamics of beliefs and knowledges of the individual that takes such a test (called from now on 'the agent').

We will use for the analysis only two questions from a simple logic game[2]:

Seven students A, B, C, D, E, F, G from an university campus are about to go to a movie. However, the students will go to the movie under the following joint conditions:

- *A and B are going to the movie, only if C is going to the movie.*[3]
- *A is going to the movie, if D is going to the movie.*
- *G is going to the movie, only if C and E are going to the movie.*
- *B is going to the movie, if G is going to the movie.*
- *If G is not going to the movie, then D is not going to the movie either.*

1. Which of the following is a complete list of students who ***could*** *be going to the movie together?*

(A) D, E, C, and B
(B) G, E, B, and C
(C) B, G, E, and D
(D) D, C, A, and G
(E) None of the above

2. If ***D*** *is going to the movie, then how many students, including D,* ***must*** *be going to the movie?*

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

[2] Adapted from Curvebreakers, *LSAT Logic Games,* 2nd Edition, New York, Chicago, San Francisco, Lisbon, London, Madrid, Mexico City, Milan, New Delhi, San Juan, Seul/Singapore, Sydney, Toronto, McGraw-Hill, 2008, p. 13.

[3] We are going to interpret both conditions as (q→p)э(r→p) rather than (qэr)→p, and (p→q)э(p→r) rather than p→(qэr). In the second interpretation the number of epistemic possible worlds is larger, yet this would not change our approach to the matter.

In answering the questions above it must be remembered that the informations given in the game are all true. For the agent in the first stage of his solving problem process only the informations given in the first two parts of the game (*fact pattern*, and *constraints*), and those that are built in the questions themselves are true. The informations embedded in the answers are, at least in the beginning of the game, *possible*.

What kinds of informations could the agent have? We can classify them as follows:

- the set of informations **given** in the game: **O** $\{f_0, f_1, \dots c_0, c_1, \dots q_0, q_1, \dots a_0, a_1, \dots\}$. The letters 'f', 'c', 'q', and 'a' stand for 'fact pattern', 'constraints', 'questions', and 'answers'. For instance, in the game above the single information given in the 'fact pattern' subclass of **O** is f_0 = "*Seven students A, B, C, D, E, F, G from an university campus are about to go to a movie*".
- the set of informations **inferred** from the informations given in the set O: **N**=$\{n_0, n_1, \dots\}$
- the set of informations given by the **rules of inference** and supposedly being part of the agent's epistemic background: **R** = $\{r_0, r_1, \dots\}$

In fact, the solving of the game could be seen as a passage from the set **O** toward the set **N** with the support of the set **R** and, certainly, with *the ability to use* this last set of informations. It is evident that the informations in the set **R** are or are not in the agent's possession. The set **N** is a dynamic one, as long as some informations will be included in the set as the game is solved.

Let us denote by letters p_i, with *i* ranging from 0 to 6, the following sentences in the game:

- p_0 = "A is going to the movie"
- p_1 = "B is going to the movie"
- p_2 = "C is going to the movie"
- p_3 = "D is going to the movie"
- p_4 = "E is going to the movie"
- p_5 = "F is going to the movie"
- p_6 = "G is going to the movie"

As the *fact pattern* of the game informs, there is a setting of seven students (information given by the game - f_0) and their participation to the movie is constrained by the following rules (written in propositional logic; this information is needed from the agent's set **R** – how to translate natural language expressions in the formal language of sentential logic – r_0):

(a) $(p_0 \rightarrow p_2)$ (c_0) & $(p_1 \rightarrow p_2)$ (c_1)
(b) $(p_3 \rightarrow p_0)$ (c_2)
(c) $(p_6 \rightarrow p_2)$ (c_3) & $(p_6 \rightarrow p_4)$ (c_4)
(d) $(p_6 \rightarrow p_1)$ (c_5)
(e) $(\neg p_6 \rightarrow \neg p_3)$ (c_6)

For each constraint, the same information from **R** (r_0) will be needed, and the procedure is reiterative, for all the cases above. Of course, we could note that the constraints from (a) to (e) are informations in the first set and are to be called c_0, c_1, c_2, c_3, c_4 c_5, and c_6 respectively, as above.

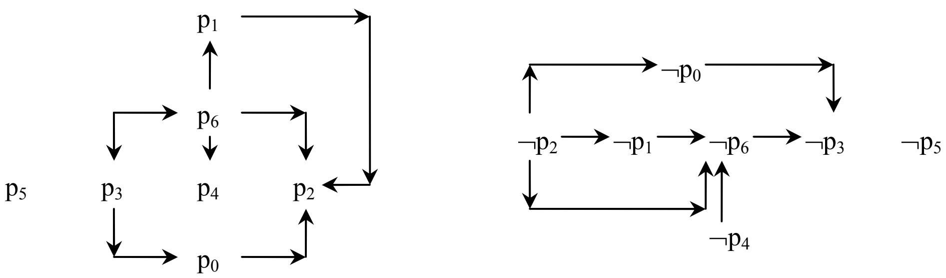
The next step in solving the game according to the standard procedure consists in writing down the contrapositives of the informations received. This should be done by the agent taking into consideration another rule from the set **R** - r_1 the contraposition rule for implications. The contrapositives are:

(f) c_0: $(p_0 \rightarrow p_2)$ - n_0: $(\neg p_2 \rightarrow \neg p_0)$
(g) c_1: $(p_1 \rightarrow p_2)$ - n_1: $(\neg p_2 \rightarrow \neg p_1)$
(h) c_2: $(p_3 \rightarrow p_0)$ - n_2: $(\neg p_0 \rightarrow \neg p_3)$
(i) c_3: $(p_6 \rightarrow p_2)$ - n_3: $(\neg p_2 \rightarrow \neg p_6)$
(j) c_4: $(p_6 \rightarrow p_4)$ - n_4: $(\neg p_4 \rightarrow \neg p_6)$
(k) c_5: $(p_6 \rightarrow p_1)$ - n_5: $(\neg p_1 \rightarrow \neg p_6)$
(l) c_6: $(\neg p_6 \rightarrow \neg p_3)$ - n_6: $(p_3 \rightarrow p_6)$

The next step is that of establishing the so called 'logic chains' in order to draw a map of the relations between all the informations available until now in the game (this procedure uses also an information from the set **R** - r_2 the relation of transitivity for material implications and an informal representation of the implications involved):

(m) From c_2: $(p_3 \rightarrow p_0)$ and c_0: $(p_0 \rightarrow p_2)$ the agent could write: n_7: $(p_3 \rightarrow p_0 \rightarrow p_2)$
(n) From c_5: $(p_6 \rightarrow p_1)$ and c_1: $(p_1 \rightarrow p_2)$ the agent could write: n_8: $(p_6 \rightarrow p_1 \rightarrow p_2)$
(o) From n_0: $(\neg p_2 \rightarrow \neg p_0)$ and n_2: $(\neg p_0 \rightarrow \neg p_3)$ the agent could write: n_9: $(\neg p_2 \rightarrow \neg p_0 \rightarrow \neg p_3)$
(p) From n_1: $(\neg p_2 \rightarrow \neg p_1)$, n_5: $(\neg p_1 \rightarrow \neg p_6)$, and c_6: $(\neg p_6 \rightarrow \neg p_3)$ the agent could write: n_{11}: $(\neg p_2 \rightarrow \neg p_1 \rightarrow \neg p_6 \rightarrow \neg p_3)$
(q) From n_6: $(p_3 \rightarrow p_6)$ and c_3: $(p_6 \rightarrow p_2)$ the agent could write: n_{10}: $(p_3 \rightarrow p_6 \rightarrow p_2)$

The informations c_4: $(p_6 \rightarrow p_4)$, n_4: $(\neg p_4 \rightarrow \neg p_6)$, and n_3: $(\neg p_2 \rightarrow \neg p_6)$ will not be lost for the chain of implications as long as the strategy of the game solving suggests that these information are to be added too to the chain, even they are not in the same line as the other implications. Therefore, the final logic map of the game could be represented as follows:



Schema (1) Logic chains of direct implication Schema (2) Logic chains of contrapositions

Figure 1. The representation of the logic map of the game

This representation of the logical relations that are simultaneously acting in the game, are in fact conjunctions of implications and contrapositions as they are either given as premises in the game or are logically derived from the premises. In fact, the second schema is a sort of contrapositive for the first one. Its importance for the solving procedure of the game is just representational, as long as it helps the agent to 'see' quicker the correct answer on it. Otherwise, it conveys not other information than the first schema.

Now the agent could step toward answering the questions fully equipped with all the necessary informations. It appears that for solving just two couple of questions in this particular game, the agent gets into too much trouble and acquires too much information than he really needs. Nevertheless, it is suggested that in a complete game all the informations offered by the game and inferred from them according to the procedure are necessary for solving the game in the allowed amount of time.

The answering process is in itself a task that requires some other informations to be used, for the language in which the inferences are to be made must be the same as the language that was used in the first part of the solving procedure. Thus, the questions and the answers should be translated from the natural language into a sort of informal language, through different pictorial representations.

1.Which of the following is a complete list of students who ***could*** *be going to the movie together?*

(A) D, E, C, and B
(B) G, E, B, and C
(C) B, G, E, and D
(D) D, C, A, and G
(E) None of the above

In order to answer the question the agent must first understand what "a complete list of students who could be going to the movie together" mean. In the informal logic approach of the game is somehow intuitive that a complete list of students that "could be going to the movie together" in the circumstances of the game has the meaning of *the longest logic chain* proposed by the answers and in accord with the first schema of the logical chains in the game (see *Figure 1* above). However, in order to decide which answer is correct, the order of appearance in the answer for each student is not unimportant. In fact, it is this particular information what will make the difference between the true/correct answer and the others. As it can be seen in the first proposed answer (A), the sentence p_3 which corresponds to the event "D is going to the movie" is followed by the sentence p_4, corresponding to the event "E in going to the movie", and such a relation is not a direct one in the first schema of the logic game. Therefore the answer (A) is false (or incorrect). The proposed answers (C) and (D) suffer from the same lack of adequacy with the logic chain schema, for there is no direct connection between the sentence p_1 ("B is going to the movie"), and sentence p_6 ("G is going to the movie"), and correspondingly between the sentence p_3 ("D is going to the movie"), and the sentence p_2 ("C is going to the movie"). The fact that those propositions are connected indirectly would count if the answer would include also the sentence from which they are connected, yet this is not the case in the proposed answers. The proposed answer (B) satisfies the conditions imposed by the game through the logic chain Schema I: there is a direct connection between p_6 and p_4, and p_1 could be true in the circumstances, because he is the consequent in the implication $p_6 \rightarrow p_1$. The same situation is applying for the implication $p_6 \rightarrow p_2$. Therefore, the fact that p_6 is the first in the logic chain to be proposed as true, entails the truth of p_4, p_1, and p_2. This condition is not respected by neither of the other proposed answers. The correct answer is, therefore, (B). There is also a complication that could not be properly solved in such a setting offered by the informal logic used in critical thinking settings. The aspect that I am referring to is the modal expression that appears in the first question, namely 'could'. What does it mean in the context of the question? As the answers suggest, it simply mean that the agent should take into consideration the one answer from the five answers proposed, because the other four answers, as the structure of the game is conceived, are surely incorrect, and therefore, 'could not' be the case. Nevertheless, is the correct answer only *a possible one* proposed from a large set of possibilities or *one of the possible ones*? We will discuss more about this in the modal treatment of this logic game.

The second question uses, again, a modal operator:

*2. If **D** is going to the movie, then how many students, including D, **must** be going to the movie?*

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

The modal operator "must" could be read as a necessary condition in an implication, a condition that is a direct consequence of the sufficient condition that it is the case that "D is going to the movie" (sentence p_3) is true. The agent has to use the rule of Modus Ponens taking into account the truth of p_3 and the truth of the premises that appear in the Schema I, and to derive the maximum number of consequences. Let us follow the agent's inquiry: if p_3 is true, and p_3 is a sufficient condition for p_6 and for p_0, it means that we are entitled to count three students that must be going to the movie: D, A, and G. Afterwards, the fact that p_0 is also true, constitutes a sufficient condition for p_2 to be true, and from p_6 being true, it also follows that p_1 must be true, as long as the implications in the Schema I are all true. The last true sentence that must be true as a consequence of p_6 being true is p_4. Therefore, we can count the following true propositions: p_3, p_0, p_6, p_1, p_2, p_4. There are six propositions that must be true in the circumstances of the question, and therefore the maximum number of six students that must be going to the movie: D, A, G, B, C, E. It follows that the correct answer to the second question is (D). The entire game could be easily solved further in the same manner.

3. What a higher logic perspective on this logic game could offer? Sure, there are some modal terms that appear in the game, yet they are all treated in a rather informal logic manner, just for the efficiency of the solving procedure. What could an analysis of the epistemic states of the agent tell us about the manoeuvres that he should do in order to solve the game?

In order to analyse the agent epistemic states and explore some of the consequences of this kind of approach to a critical thinking game setting, we must summarise the theoretical backgrounds of such an enterprise. First of all we will take into consideration only propositional modal logic, as it is a sufficient and a necessary background for the setting of this logic game. Then we will try to build up a model for this particular game, according to Kripke semantics. We will not enter into all the theoretical details of such an approach. We will rather try to shortly describe what appears to be the basics of it.

In order to describe the epistemic states of the agent we will start to use a language for epistemic formulas[4]:

- A set **P** of propositional constants (atoms), for this particular case P = $\{p_0, p_1, p_2, p_3, p_4, p_5, p_6\}$, with p_i = 'X goes to the movie';
- One epistemic agent
- A set L_k (P) of epistemic formulas φ, ψ, ..., as the smallest set closed under:
  - If $p\in P$, then $p\in L_k(P)$
  - $\varphi, \psi \in L_k(P)$, then $(\varphi\&\psi), \neg\varphi\in L_k(P)$
  - If $\varphi\in L_k(P)$, then $K\varphi\in L_k(P)$, where Kφ is read as: 'the agent knows that φ'.

We can now define a Kripke Model *M*. By definition, a Kripke model M is a triple ⟨W, V, R⟩, where:

- ***S*** is a non-empty set of *epistemic states* $\{s_0, s_1, s_2, \ldots, s_i, \ldots, s_j, \ldots, s_n\}$
- ***V***: $\boldsymbol{S} \rightarrow (\boldsymbol{P}\rightarrow\{\mathbf{t}, \mathbf{f}\})$ is a truth assignment to the propositional atoms per state
- $\boldsymbol{R}\subseteq \boldsymbol{S} \times \boldsymbol{S}$ is the set of possibility/accessibility relations

The interpretation of the accessibility relation $(\mathbf{s_i},\mathbf{s_j})\in\boldsymbol{R}$ is understood as follows: in a world **w** defined by the model **M** and a state **s** the agent considers the world (M,s) as a possible world, or an *epistemic alternative*, on the basis of his knowledge. Kripke semantics of epistemic formulas define a relation $w\vDash\varphi$ (φ is true in *w*) thus:

(1) $(M, s)\vDash p \Leftrightarrow V(s)(p) = t$ for $p\in P$

(2) $(M, s)\vDash\varphi\wedge\Psi \Leftrightarrow (M, s)\vDash\varphi$ and $(M, s)\vDash\Psi$

(3) $(M, s)\vDash\neg\varphi \Leftrightarrow (M, s)\nvDash\varphi$

(4) $(M, s)\vDash K\varphi \Leftrightarrow (M, s)\vDash\varphi$ for all t with $(s, t)\in R$

[4] J.-J. Ch Meyer, W. van der Hoek, *Epistemic logic for AI and Computer Science*, Cambridge, New York, Melbourne, Madrid, Cape Town, Cambridge University Press, 1995, pp. 8-9.

The modal epistemic clause states that "the agent knows φ in a world (M, s), iff φ is true in all the worlds that the agent considers possible. In the epistemic state ***s***, the agent has doubts about the true nature of the real world, so he considers several worlds ***t*** as possible, as the R(*s*,t) holds. If all epistemic alternatives with R(s,t) holds that φ, then the agent Kφ.

There are some remarks to be made in this point. As long as the logic used in the game is classic propositional logic, we can safely talk about modal propositional logic (MPL) in this new setting, without changing any requests that interests the solving procedure, or the set of formulae in the set **P**. Therefore we must note that each model of MPL is in fact composed by a frame ⟨***S*, *R***⟩ and a function ***V*** over that frame, that specifies the truth values of the formulae in each element of the set **P**. The frame contains informations about how many worlds are and which worlds are accessible from which. The underlying structure of the frame is that of a directed graph[5] in which the binary relation ***R*** is describing the accessibility relation between the elements of ***S*.** The accessibility relation has the formal feature required by the modal logical system[6]: no accessibility relation required in the modal system **K**, a serial one over ***S*** in the system D, a reflexive relation over ***S*** in the systems **T**, **B**, $\mathbf{S_4}$ and $\mathbf{S_5}$, a symmetric one in **B**, and $\mathbf{S_5}$, and transitive relation in $\mathbf{S_4}$ and $\mathbf{S_5}$[7]. What could be the accessibility relations in our setting? In order to answer to this question we should take a look at the informations that the game offers and requires for its solving. In the set **O**, and **R** described in the §2, we have only propositional logic formulae, the rule of Modus Ponens, and the rule for transitivity for material implications. It seems that as long as the setting of the game will use only propositional logic formulae, there will be no real need for a stronger modal system and correspondingly for stronger accessibility relations. Yet we want to endow the epistemic agent with the ability to 'see' all the epistemic possibilities of the game and therefore we will make a decision for total accessibility relations between the worlds.

Also, it seems clear that, as long as the model requires, the sets of informations of the game **O**, **N**, **R** (see §2) are distributed as follows: the informations from the set **R** will count as formulae in the set ***P*** and, according to the modal system adopted, part of the modal system formulae as well, for it all includes propositional logic tautologies. The informations in the sets **O** and **R** are to be informations that could be introduced into the set ***P*** via what could be seen as an information updating process. The same interpretation is useful for the question and answers information intake.

[5] A directed graph is a set of vertices $W = \{w_0, w_1, ..., w_i, ...\}$, with *i* ranging from 0 to n, and a set of edges $R = \{(w_0,w_1), (w_0, w_2), ..., (w_i,w_j), ...\}$ directed from the source vertex w_i to the target vertex w_j, with $w_i, w_j \in W$, and i, j ranging from 0 to n. The set R is a subset of the cross product WxW of ordered pairs, and is a binary relation over W.

[6] Theodore Sider, *Logic for Philosophy*, New York: Oxford University Press, 2010, p. 161.

[7] We are not to discuss here the modal systems axioms, rules of derivation and their relations. It is known that Modus Ponens is a rule of derivation in all of them, and also all tautologies in propositional logic are axioms of any of them. For our purpose here this is all that is required for the solving procedure. The rest of the setting, the modal system, the epistemic axioms included, the accessibility relations are to be mentioned as a logic frame for the analysis when the case.

We want to be more specific about the accessibility relations defined in the model. What kind of relations the structure of the game necessitates? It should be taken into account that the subjacent directed graph that constitutes the frame of the Kripke model is determined by a set ***S*** of vertices (in our case the set of possible worlds or epistemic states) and their accessibility relation. Let us start with a simpler example in order to clarify this aspect. Let the set of possible states S be composed of three elements $\{s_0, s_1, s_2\}$ and the accessibility relation over S, R, is as follows $\{(s_0, s_1), (s_0, s_2)\}$ and V $(s_0, p) = 1$ $V(s_1, p) = 1$, and $V(s_2, p) = 0$. We have therefore a Kripke model as pictured bellow (*Figure 2*).

Then, what can we say about the Kp in s_0? Its truth value could be very easily inferred from the definition, for as we have written above $(M, s)\vDash K\varphi \Leftrightarrow (M, s)\vDash\varphi$ for all t with $(s, t)\in R$, means that all the possible states to which the agent has access from s_0 (in this case, according to the frame, s_1 and s_2), must contain a valuation 1 for the **p**. Yet in s_2 the value for **p** is false (0), and therefore K**p** in s_0 is also false. The same applies for K¬**p** in s_0. We will add to the semantics of the epistemic formulas in Kripke model another operator Bφ which will be read as "the agent believes p", when belief is thought as possibility. Thus, in the model M ⟨S, V, R⟩ above we also have:

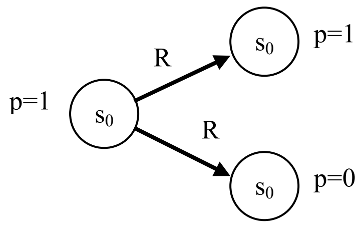


Figure 2. Example 1 of Kripke model and valuation of formulae

(5) $(M, s)\vDash B\varphi \Leftrightarrow (M, s)\vDash\varphi$, iff $\exists t, (s, t)\in R$ and $(M, t)\vDash \varphi$

In the *Figure 2*, the valuation of B**p** is true, because there is at least a state (s_1) or possible epistemic alternative for the agent, in which the valuation for **p** is true.

However, if we change the frame of the epistemic model, that is the number of epistemic states and/or the set R of the accessibility relations, the valuations of the formulae will also change. In the new model depicted in *Figure 3* bellow, the agent doesn't know **p** (Kp = 0) in the same state s_0, for there still aren't accessible epistemic states where all the values of **p** are true (the epistemic state available from s_0 contains a false value for **p**), yet the agent also doesn't believe **p** (Bp = 0) for there is no accessible epistemic state with **p**=1.

Therefore, even if we take into consideration the weakest epistemic modal logic system (K) for the needs of the game, still, the frame of the model could play an important evaluative function. Apart from these important changes in valuation, there are some other philosophical and logical issues in the way knowledge and belief could be realistically modelled. For instance, *the problem of omniscience*, which treats some forms of epistemic belief as being

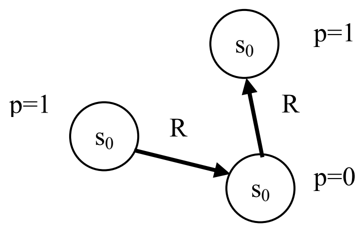


Figure 3. Example 2 of Kripke model and valuation of formulae

closed under of number of modal epistemic properties that are rather idealistic[8]. We will depict only two of them that could have a direct influence on our analysis:

Belief of valid formulas: $\vDash\varphi \Rightarrow \vDash B\varphi$
Closer under valid implication: $\vDash\varphi\rightarrow\psi \Rightarrow \vDash B\varphi\rightarrow B\psi$

As Meyer and vand der Hoek shows, there is a trial to solve the logical omniscience problem by distinguishing implicit and explicit belief. By defining implicit belief (B), an ideal form of belief, as a belief that agent may be unaware of, and explicit belief (B_e) as a belief of wich the agent is aware of. A model for B_e was proposed in which the formal semantics of formulas are defined in the so-called *partial possible world*s or *situations*. However, if we considered important to note that there could be complications for interpreting epistemic modalities, we will not dwell into modeling the present logic game with such a frame here.

Let us begin with the first sentence of the game: The first sentence of the game offers the agent the total number of possible worlds that could be developed from this setting: seven propositions (students) that could be arranged in 2^7 ways, ranging from a world s_0, where all the propositions are false, to a world s_{127} that contains all the true propositions. We have, therefore, a set of possible worlds *S* ($s_0, s_1, \ldots., s_{127}$). Therefore, in the beginning of the game, the agent knows on the basis of the first sentence given in the game, that there are 128 possible worlds for the setting of this game. So, his epistemic state is described by the model of all this possible worlds or epistemic alternatives[9]. This worlds and the valuation for each world of the atomic propositions in the game is depicted in the *Figure 4* bellow.

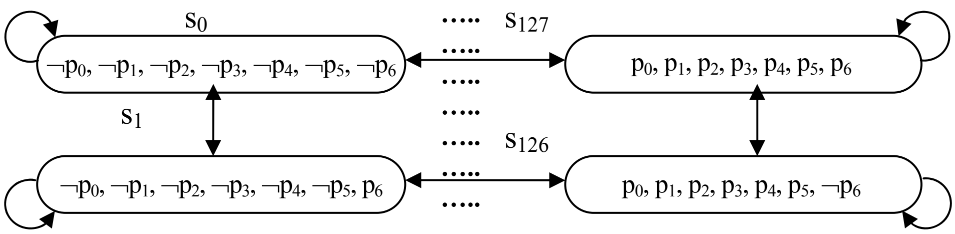


Figure 4. The initial epistemic model of the game for one agent

After receiving the informations from the constraints of the game ($c_0 \ni c_1$, c_2, $c_3 \ni c_4$, c_5, and c_6) the agent will drop the possible worlds (or give up the epistemic states) that won't meet the *joint conditions* of those informations (the epistemic states that are impossible in the conditions of the new informations received). It is easy to verify that for each information received by the agent as he updates his set of formulae, there is an updated epistemic model for the agent states which correspond to a successive reduction of the initial epistemic model and all these

[8] J.-J. Ch Meyer, W. van der Hoek, *op. cit.*, p. 74.
[9] J. van Benthem, *Modal Logic for Open Minds*, Center for the Study of Language and Information, 2010, p. 118.

models are included in the initial model[10]. The amount of successive worlds included in the successive models are containing 80, 56, 40, 34, and finally 26 epistemic possible worlds. The last model's worlds are depicted bellow as a table in *Table 1*: The remaining Possible Worlds (PW) after the first set of informational import for the epistemic agent and the corresponding sentence valuation (V) on each PW.

Table 1.

Worlds/ Propositions	p_0	p_1	p_2	p_3	p_4	p_5	p_6
w_0	0	0	0	0	0	0	0
w_2	0	0	0	0	0	**1**	0
w_4	0	0	0	0	**1**	0	0
w_6	0	0	0	0	**1**	**1**	0
w_{16}	0	0	**1**	0	0	0	0
w_{18}	0	0	**1**	0	0	**1**	0
w_{20}	0	0	**1**	0	**1**	0	0
w_{22}	0	0	**1**	0	**1**	**1**	0
w_{48}	0	**1**	**1**	0	0	0	0
w_{50}	0	**1**	**1**	0	0	**1**	0
w_{52}	0	**1**	**1**	0	**1**	0	0
w_{53}	0	**1**	**1**	0	**1**	0	**1**
w_{54}	0	**1**	**1**	0	**1**	**1**	0
w_{55}	0	**1**	**1**	0	**1**	**1**	**1**
w_{80}	**1**	0	**1**	0	0	0	0
w_{82}	**1**	0	**1**	0	0	**1**	0
w_{84}	**1**	0	**1**	0	**1**	0	0
w_{86}	**1**	0	**1**	0	**1**	**1**	0
w_{112}	**1**	**1**	**1**	0	0	0	0
w_{114}	**1**	**1**	**1**	0	0	**1**	0
w_{116}	**1**	**1**	**1**	0	**1**	0	0
w_{117}	**1**	**1**	**1**	0	**1**	0	**1**
w_{118}	**1**	**1**	**1**	0	**1**	**1**	0
w_{119}	**1**	**1**	**1**	0	**1**	**1**	**1**
w_{125}	**1**	**1**	**1**	**1**	**1**	0	**1**
w_{127}	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Still, for the sake of the epistemic model, there is no necessity to draw all these possible epistemic states and an accessibility relation between them. We could build a simpler Kripke model that will satisfy our joint condition from the conjunction of implications (see above) and using an inverse procedure that one proposed by Sider[11] or following the general lines of building a Kripke model from Gasquet[12]. We should first transform the true formula in the game (it was taken as a true hypothesis for solving the game) according to the rules of classical propositional logic:

$$(p_0 \rightarrow p_2) \wedge (p_1 \rightarrow p_2) \wedge (p_3 \rightarrow p_0) \wedge (p_6 \rightarrow p_2) \wedge (p_6 \rightarrow p_4) \wedge (p_6 \rightarrow p_1) \wedge (\neg p_6 \rightarrow \neg p_3)$$

becomes

$$(\neg p_0 \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge (\neg p_3 \vee p_0) \wedge (\neg p_6 \vee p_2) \wedge (\neg p_6 \vee p_4) \wedge (\neg p_6 \vee p_1) \wedge (p_6 \vee \neg p_3)$$

We will consider this last equivalent formula the formula that is true in the initial epistemic state of the agent and we will try to draw the model from here. The main idea in constructing the model starting with a formula is to take into account the truth conditions for the formula (see (2) to (5) above in the semantics of the model). First, as the Gasquet at al. recommends, there should be drawn a world for the initial state in which the formula is given (*Figure 5*). Then, according to the rule for conjunction (see (2) for the model's

[10] *Ibidem*.
[11] Th. Sider, *op. cit.*, p. 183.
[12] Gasquet, Olivier, Andreas Herzig, Bilal Said, François Schwarzentruber, *Kripke's Worlds. An Introduction to Modal Logics via Tableaux*, Basel, Heidelberg, New York, Dordrecht, London, Springer, 2014, pp. 53-60.

requirements for conjunction formulae), the formulae composing the conjunction should be separated and written one under the other (*Figure 6*).

s_0 $(\neg p_0(p_2)\wedge(\neg p_1\vee p_2)\wedge(\neg p_3\vee p_0)\wedge(\neg p_6\vee p_2)\wedge(\neg p_6\vee p_4)\wedge(\neg p_6\vee p_1)\wedge(p_6\vee\neg p_3)$

Figure 5. The representation of the formula in the initial epistemic state of the model

Because a disjunction is satisfied when at least one of its component formulas is satisfied, it is recommended that for each component in the disjunction to create a pre-model of the initial model[13].

$\neg p_0\vee p_2$
$\neg p_1\vee p_2$
$\neg p_3\vee p_0$
$\neg p_6\vee p_2$
$\neg p_6\vee p_4$
$\neg p_6\vee p_1$
$p_6\vee\neg p_3$

s_0

Figure 6. The representation of the formula in the initial epistemic state of the model after the first rule for conjunction

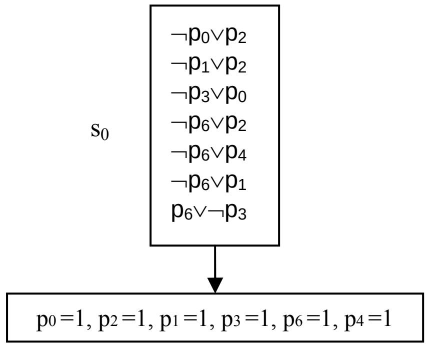


Figure 7. The representation of one premodel

In the *Figure 7* above, the representation of the pre-model includes just a single possible epistemic state s'_1 which is sufficient to prove that the agent knows in the epistemic state s_0 that the formula holds, for in any accessible possible state the formula holds. Yet, this is not the single possible pre-model to be drawn. In fact it important in this step of the procedure to process all the possible pre-models of the game that satisfies the formula.

After combining all the pre-models and eliminating those that contain simultaneously p_i and $\neg p_i$ in some epistemic states, we would find a situation similar with the one depicted in *Table 1*. In the boxes representing the epistemic states from the *Figure 8* bellow there were written the valuations for each atomic formula that formed the initial formulas in the order of their indexes. However, in the present state of the model, which contains a complete depiction of the epistemic states after the import of the informations in the

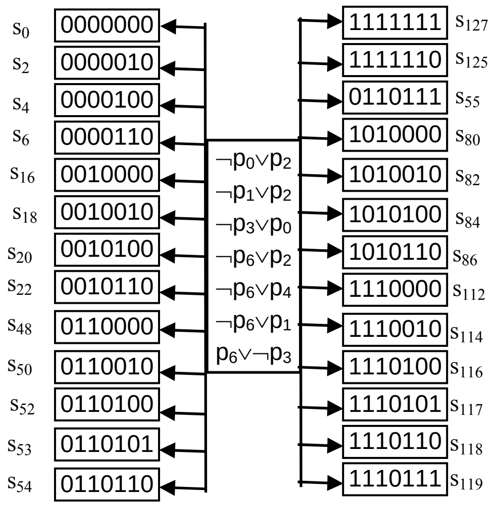


Figure 8. The representation of the model

[13] *Ibidem*, p. 58.

constraints of the game, we could be made some remarks. The agent knows the formula to be true in all possible states accessible to him, yet he does not know in which epistemic state he is. The model could become more adequate by adding an accessibility relation from each state to itself. And, if we want that the epistemic states to be symmetrical (meaning that the agent could start the game from any particular state with the same chance of success), then the accessibility relations should be drawn accordingly.

We must now extract the Kripke model and check it before we could pass to some other considerations on the epistemic states of the agent and furthermore, to try to see if it is a sufficiently elaborate model to answer the questions in the game. We have a model $\mathbf{M} = \langle \boldsymbol{S}, \boldsymbol{R}, \boldsymbol{V} \rangle$ such that:

- $\boldsymbol{S} = \{s_0, s_1, s_2, s_3, \ldots, s_{24}, s_{25}\}$
- $\boldsymbol{R} = \{(s_0, s_1), (s_0, s_2)\ (s_0, s_3) \ldots (s_0, s_{24})\ (s_0, s_{25})\}$
- $\boldsymbol{P} = \{p_0, p_1, p_2, p_3, p_4, p_6\}$
- And we have also a valuation $\boldsymbol{V}$ in each epistemic state and illustrated on the model representation.

It seems that the model has all the characteristics of a Kripke model.

If we want to check if the formula holds for every epistemic state available, to see if the agent knows (r), then we should verify if the formula is true in all epistemic states available. And the formula is true as the model was constructed this way. More than that, the agent knows the formula (Kφ) or in a weaker perspective on his epistemic states, he also believes the formula (Bφ).

Let us try now to answer to the first question in the game:

1.Which of the following is a complete list of students who ***could*** *be going to the movie together?*

(A) D, E, C, and B
(B) G, E, B, and C
(C) B, G, E, and D
(D) D, C, A, and G
(E) None of the above

As it was shown in the classic procedure for solving the game, the answers should be reformulated in order to be further analysed in this frame. As it was noticed, each answer represents in fact a series of applications of the Modus Ponens rule. For in the case of (A), given the fact that D will go to the movie (p_3 is true) is will E have go to the movie (p_4 must be true)? According to the initial formula there is no such inference, because the needed premise $p_3 \rightarrow p_4$ is missing from the set **P** of formulae of the model. As in the classic solving procedure of the game there is only one possible answer: (B). However, the number of possible epistemic states in which this answer is true is six: s_{53}, s_{55}, s_{117}, s_{119}, s_{125}, s_{127}. So, the agent has good reason to believe that the answer (B) is true (meaning that the formula associated with the answer is true in some possible worlds in the model), yet it can't be said that the agent knows the truth of (B). For, according to the model, the formula for (B) is not true in all possible worlds in the model.

The situation is similar for the second question of the game. For in order to count the number of the maximum students that will go to the movie if a certain condition is true ("D is going to the movie") the agent should try to see in which world such formula will be true. And the case is that the only epistemic state or possible world that make the formula true is s_{125}. In fact, the series of deductive inferences made with the Modus Ponens rule is six. Again, as the formula is true in a single possible epistemic state (including the state itself) the agent just believes the answer to be true.

4. Some developments of the new setting for logic games are at hand. For instance, the game could be easily solved with approximatively the same easiness as in the classical procedure, yet the possibilities for diversifying the frame and the types of questions in the game are great. Instead of developing different kinds of relations between a large number of objects in the game (seven students, for this game will lead to a large initial number of epistemic states/possible worlds scenario) in order to test the analytical ability of an individual just for a particular type of thinking that involves only propositional logic, lowering the number of objects and introducing the modal type of approach to the game would offer the possibility to test the ability to think critically in complex social situations. Introducing questions concerning the epistemic status of an agent in such and such world in a particular model, or asking questions about the epistemic states of multiple agents interacting in different types of games like the games of cards[14] would make a difference. Even a particular setting for a game can be diversified using a different frame for the same formulae given. More, the possibilities for change of the new general setting could receive influence from other types of logical approaches to social interaction, as depicted in game theory, the dynamic logic for actions and events, and so on[15].

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[14] J. van Benthem, *op. cit.*, p. 136.
[15] *Ibidem*, pp. 127-218.